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Axial Velocity Solution for Spinning-Up Rigid Bodies Subject to Constant Forces

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An approximate analytical solution is presented for the axial velocity of a spinning-up spacecraft subject to constant body-fixed forces and moments. We assume that the rigid body spins about the maximum or the minimum principal moment of inertia. Also, it is assumed that the deviation of the spin axis from an inertially fixed direction is small. Our closed-form solution for axial velocity applies to axisymmetric, nearly axisymmetric, and in some cases asymmetric rigid bodies. Furthermore, asymptotic limits are determined for large spin rates and for some geometric limits such as a sphere, a thin rod, and a flat disk. When the axial body-fixed force is nonzero, the axial velocity is dominated by the expected secular term of axial acceleration multiplied by time. However, when the axial force is zero, then the much more complicated effects of the transverse forces come into play: there are no longer any secular terms, but rather there is an asymptotic limit to the axial velocity accrued from the projection of the transverse accelerations onto the inertial direction of the initial spin axis. The axial velocity solution consists of Fresnel integrals, integrals of Fresnel integrals, and many new families of related integrals. Numerical simulation demonstrates the accuracy of the analytical solution in a practical example.

Nomenclature

A	=	direction cosine matrix
A_v	=	fundamental integrals for axial velocity solution
c	=	cosine
E	=	complex Fresnel integral
E_i	=	exponential integral
F_x, F_y	=	scaled transverse angular acceleration
f_x, f_y, f_z	=	body-fixed forces, N
H	=	angular momentum, $\mathrm{kg \cdot m^2/s}$
I_{u0}, J_u	=	fundamental integrals for angular velocity solution
I_x, I_y, I_z	=	principal moments of inertia, $\mathrm{kg \cdot m^2}$
I_ϕ	=	fundamental integral for Euler angle solution
i	=	$\sqrt{-1}$ (imaginary number)
k, k_x, k_y	=	dimensionless moment-of-inertial parameters
M_x, M_y, M_z	=	body-fixed moments, $\mathrm{N \cdot m}$
m	=	mass, kg
s	=	sine
sgn	=	signum function
t	=	time, s
v	=	linear velocity, m/s
XYZ	=	inertial frame
xyz	=	body-fixed frame
λ	=	inverse axial angular acceleration, $\mathrm{s^2}$
ρ_0	=	angular momentum pointing error angle in inertial frame, rad
τ	=	spin rate, rad/s
ϕ_x, ϕ_y, ϕ_z	=	Eulerian angles, rad
Ω	=	scaled angular velocity, rad/s
ω	=	angular velocity, rad/s

Subscript

sec	=	secular solution

Superscripts

$\dot{}$	=	time derivative in inertial frame
$\bar{}$	=	complex conjugate

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Introduction

ANALYTICAL solutions for rigid-body motion and for rocket and spacecraft maneuvers have a rich history [1–36]. Most of these analytical solutions are devoted to the rotational motion and only a few introduce closed-form solutions (in special cases) for the velocity problem of a spinning rigid body [30–32].

Ayoubi and Longuski [34] introduced approximate closed-form solutions for transverse velocity and displacement of a spinning-up rigid body when body-fixed forces and moments are constant. In this paper, we use the results of Ayoubi and Longuski [35] and Ayoubi [37] to find the analytical solution for the axial velocity of a spinning-up rigid body. We also provide the asymptotic limits by identifying secular terms (when they exist) and by determining limits (as time goes to infinity) in the analytical solution. When two of the Eulerian angles are small enough, the solution is valid for axisymmetric, nearly axisymmetric, and under special conditions, for asymmetric rigid bodies.

Euler's Equations of Motion and Kinematic Equations

Figure 1 depicts a spinning rigid body subjected to body-fixed forces (f_x, f_y, f_z) and body-fixed moments (M_x, M_y, M_z). The body spin axis is assumed to be the z axis and because there is an assumed moment about this axis, the body will spin up or down.

We assume throughout that the body-fixed moments are constant. For axisymmetric, nearly axisymmetric rigid bodies, or asymmetric rigid bodies where the product $(I_x - I_y)/I_z\omega_x\omega_y$ is small enough compared to M_z/I_z, Euler's equations of motion [38] can be simplified as

$$\dot{\omega}_x(t) = M_x/I_x - [(I_z - I_y)/I_x]\omega_y\omega_z \tag{1}$$

$$\dot{\omega}_y(t) = M_y/I_y - [(I_x - I_z)/I_y]\omega_z\omega_x \tag{2}$$

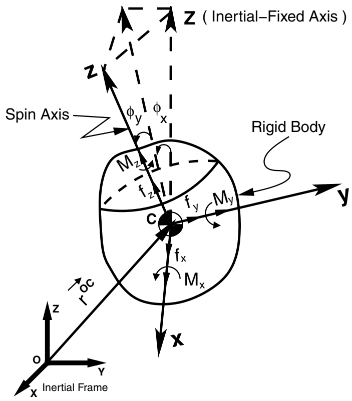

Fig. 1 A model for a spinning-up rigid body in an inertial frame.

$$\dot{\omega}_z(t) \approx M_z/I_z \tag{3}$$

By integrating Eq. (3) and assuming that axial moment M_z is constant, we obtain

$$\omega_z(t) \approx (M_z/I_z)t + \omega_{z0}, \qquad \omega_{z0} \triangleq \omega_z(0) \tag{4}$$

which is, of course, exact for axisymmetric rigid bodies.

A Type I: 3-1-2 Euler angle sequence [39] is used to relate the orientation of a body-fixed reference frame to an inertial reference frame.

With the assumptions that ϕ_x and ϕ_y are small, which means the excursion of the angular momentum vector H, from the inertial XY plane is small (a reasonable assumption for a spin-stabilized spacecraft), and that $\phi_y\omega_x$ is small compared to ω_z, the kinematic equations can be simplified as

$$\dot{\phi}_x = \omega_x + \omega_z\phi_y \tag{5}$$

$$\dot{\phi}_y = \omega_y - \phi_x\omega_z \tag{6}$$

$$\dot{\phi}_z = \omega_z \tag{7}$$

After substituting Eq. (7) into Eq. (3) and integrating, we obtain

$$\phi_z = \frac{1}{2}\frac{M_z}{I_z}t^2 + \omega_{z0}t + \phi_{z0}, \qquad \phi_{z0} \triangleq \phi_z(0) \tag{8}$$

Closed-Form Analytical Solution for Angular Velocities

Using the results of Tsiotras and Longuski [29], the closed-form solutions for scaled angular velocities Ω_x and Ω_y can be written as

$$\Omega(\tau) = \Omega(\tau_0)e^{i\rho(\tau^2-\tau_0^2)/2} + Fe^{i\rho\tau^2/2}\bar{I}_{u0}(\tau_0,\tau;\rho) \tag{9}$$

where the overbar denotes a complex conjugate and

$$\Omega \triangleq \Omega_x + i\Omega_y \tag{10}$$

where

$$\Omega_x \triangleq \omega_x\sqrt{|k_y|}, \qquad \Omega_y \triangleq \omega_y\sqrt{|k_x|} \tag{11}$$

$$k_x \triangleq \frac{I_z - I_y}{I_x}, \qquad k_y \triangleq \frac{I_z - I_x}{I_y} \tag{12}$$

In Eq. (11), ω_x and ω_y are the angular velocities in the body-fixed frame; F in Eq. (9) is defined as

$$F \triangleq F_x + iF_y \tag{13}$$

where

$$F_x \triangleq (M_x/I_x)(I_z/M_z)\sqrt{|k_y|} \tag{14}$$

$$F_y \triangleq (M_y/I_y)(I_z/M_z)\sqrt{|k_x|} \tag{15}$$

The function $I_{u0}(\tau_0,\tau;\rho)$ in Eq. (9) is defined as

$$I_{u0}(\tau_0,\tau;\lambda) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\,\mathrm{d}u = I_{u0}(\tau;\lambda) - I_{u0}(\tau_0;\lambda) \tag{16}$$

where

$$\lambda \triangleq I_z/M_z \tag{17}$$

$$\rho \triangleq k(I_z/M_z) \tag{18}$$

and

$$k \triangleq \sqrt{k_x k_y} \tag{19}$$

$$\tau(t) \triangleq \omega_z(t) = (M_z/I_z)t + \omega_{z0}, \qquad \omega_{z0} = \tau_0 \tag{20}$$

Closed-Form Analytical Solution for Eulerian Angles

With the assumption that ϕ_x, ϕ_y, and $\phi_y\omega_x$ are small, Tsiotras and Longuski [29] showed that for the Euler angle sequence 3-1-2 (ϕ_z,ϕ_x,ϕ_y), the closed-form solution is

$$\phi(\tau) = \phi(\tau_0)e^{-i\lambda(\tau^2-\tau_0^2)/2} + \lambda e^{(-i\lambda\tau^2/2)}I_\phi(\tau_0,\tau;\lambda,\rho) \tag{21}$$

where

$$\phi = \phi_x(t) + i\phi_y(t) \tag{22}$$

$$I_\phi(\tau_0,\tau;\lambda,\rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\omega(u)\,\mathrm{d}u \tag{23}$$

$$I_\phi(\tau_0,\tau;\lambda,\rho) = k_1 I_{\phi1}(\tau_0,\tau;\lambda,\rho) + k_2 I_{\phi2}(\tau_0,\tau;\kappa,\rho) \tag{24}$$

and

$$k_1 \triangleq (\sqrt{|k_x|} + \sqrt{|k_y|})/2k, \qquad k_2 \triangleq (\sqrt{|k_x|} - \sqrt{|k_y|})/2k \tag{25}$$

$$\rho = k\lambda, \qquad \mu \triangleq \lambda + \rho = \lambda(1+k), \qquad \kappa \triangleq \lambda - \rho = \lambda(1-k) \tag{26}$$

where the ks represent the mass properties of the rigid body, and $I_{\phi2}$ provides the contribution from an asymmetric body. The integrals $I_{\phi1}$ and $I_{\phi2}$ are evaluated as follows:

$$I_{\phi1}(\tau_0,\tau;\lambda,\rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\Omega(u)\,\mathrm{d}u \tag{27}$$

$$I_{\phi2}(\tau_0,\tau;\kappa,\rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\bar{\Omega}(u)\,\mathrm{d}u \tag{28}$$

$$I_{\phi1}(\tau_0,\tau;\lambda,\rho) = [\Omega(\tau_0)e^{(-i\rho\tau_0^2/2)} - F\bar{I}_{u0}(\tau_0;\rho)]\bar{I}_{u0}(\tau_0,\tau;-\mu) + FJ_{u0}(\tau_0,\tau;\mu,\rho) \tag{29}$$

$$I_{\phi2}(\tau_0,\tau;\kappa,\rho) = [\bar{\Omega}(\tau_0)e^{(i\rho\tau_0^2/2)} - \bar{F}I_{u0}(\tau_0;\rho)]I_{u0}(\tau_0,\tau;\kappa) + \bar{F}\bar{J}_{u0}(\tau_0,\tau;-\kappa,\rho) \tag{30}$$

$$J_{u0}(\tau_0, \tau; \mu, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\mu u^2/2)} \bar{I}_{u0}(u; \rho)\, \mathrm{d}u \tag{31}$$

Appendix A provides an approximation for the integral $J_{u0}(\tau_0, \tau; \mu, \rho)$.

Inertial Acceleration Equation

Let us consider a rigid body which is subjected to constant body-fixed forces f_x, f_y, and f_z as shown in Fig. 1. The following equation relates the acceleration in the body-fixed and inertial frames:

$$\begin{Bmatrix} \dot{v}_X(t) \\ \dot{v}_Y(t) \\ \dot{v}_Z(t) \end{Bmatrix} = [A]_{312} \begin{Bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{Bmatrix} \tag{32}$$

where $[A]_{312}$ is the direction cosine matrix that for a 3-1-2 Euler sequence is given by [39]

$$[A]_{312} = \begin{bmatrix} c\phi_z c\phi_y - s\phi_z s\phi_x s\phi_y & -s\phi_z c\phi_x & c\phi_z s\phi_y + s\phi_z s\phi_x c\phi_y \\ s\phi_z c\phi_y + c\phi_z s\phi_x s\phi_y & c\phi_z c\phi_x & s\phi_z s\phi_y - c\phi_z s\phi_x c\phi_y \\ -c\phi_x s\phi_y & s\phi_x & c\phi_x c\phi_y \end{bmatrix} \tag{33}$$

When ϕ_x and ϕ_y are small, the direction cosine matrix can be simplified as

$$[A]_{312} \approx \begin{bmatrix} c\phi_z & -s\phi_z & \phi_y c\phi_z + \phi_x s\phi_z \\ s\phi_z & c\phi_z & \phi_y s\phi_z - \phi_x c\phi_z \\ -\phi_y & \phi_x & 1 \end{bmatrix} \tag{34}$$

Because the angle between the inertial Z and the body-fixed z is small, the last row of Eq. (32) can be written in the following form:

$$\dot{v}_Z(t) = f_z/m + (i/2m)[\bar{f}\phi(t) - f\bar{\phi}(t)] \tag{35}$$

where we note that the axial acceleration (to a high degree of accuracy) is equal to the inertial Z acceleration.

Behavior of Angular Momentum Vector

In 1989, Longuski [26] showed that for a spinning-up rigid body, the tip of the angular momentum vector (initially aligned with the inertial Z axis) traces a spiral in inertial space (as shown in Fig. 2) with a pointing error angle ρ_0 in the inertial frame which is given by

$$\rho_0 \approx \frac{\sqrt{M_x^2 + M_y^2}}{I_z \omega_{z0}^2} \tag{36}$$

when ρ_0 is small (i.e., $\rho_0 \ll 1$). When the transverse forces are zero and the axial force f_z is a nonzero constant, then for an axially symmetric rigid body [with zero initial conditions $\omega_x(0) = \omega_y(0) = 0$, $\phi_x(0) = \phi_y(0) = \phi_z(0) = 0$] we have the following secular terms [28]:

$$\Delta v_X = -\frac{f_z}{m} \frac{M_y}{I_z \omega_{z0}^2} t \tag{37}$$

$$\Delta v_Y = \frac{f_z}{m} \frac{M_x}{I_z \omega_{z0}^2} t \tag{38}$$

$$\Delta v_Z = \frac{f_z}{m} t \tag{39}$$

which are directly related to the angular momentum pointing error ρ_0 of Eq. (36). We note that by dividing Δv_X and Δv_Y by Δv_Z and taking the square root of the sum of the squares, we obtain Eq. (36).

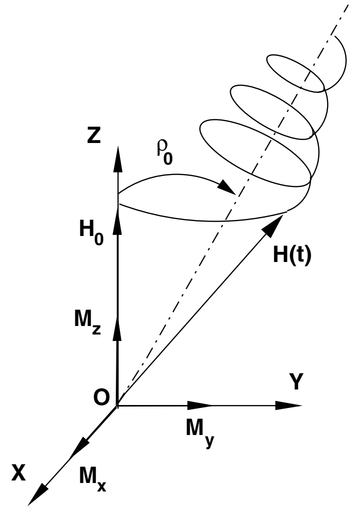

Fig. 2 Qualitative behavior of angular momentum vector in the inertial frame (based on Longuski et al. [26]). The body-fixed moments are initially coincident with the inertial axes.

Closed-Form Analytical Solution for Axial Velocity

Before integrating Eq. (35), we find it useful to replace the variable t with τ. By multiplying both sides of Eq. (37) by $\mathrm{d}t/\mathrm{d}\tau$ and using Eq. (20), we obtain

$$\frac{\mathrm{d}v_Z}{\mathrm{d}\tau} = \frac{\lambda f_z}{m} - \frac{\lambda}{m} \mathrm{Im}[\bar{f}\phi(\tau)] \tag{40}$$

where Im[·] stands for the imaginary part. After integration of both sides of Eq. (40) with respect to τ, we get

$$v_Z(\tau) = v_Z(\tau_0) + \frac{\lambda f_z}{m}(\tau - \tau_0) - \frac{\lambda}{m} \mathrm{Im}[A_{v11}(\tau_0, \tau; \lambda)] \tag{41}$$

where A_{v11} is defined as

$$A_{v11}(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} \bar{f}\phi(u)\, \mathrm{d}u \tag{42}$$

On the right-hand side of Eq. (41), the second term represents the contribution of the axial body-fixed force (f_z) as seen earlier in Eq. (39). The third term represents the contribution of the transverse body-fixed forces (f_x, f_y) on the axial velocity solution. For the thrusting case, when $f_x = f_y = 0$ and $f_z \neq 0$, the velocity changes linearly with the new independent variable τ, as we expect. The velocity solution is much more complicated when the transverse forces are nonzero (f_x, $f_y \neq 0$). In this case the axial velocity solution is dependent on many terms which derive from the Euler angle solution with no particular term being dominant. Thus, it appears that we cannot select a few terms from the analytical solution to obtain a good approximation.

We will present a complete analytical solution for the general case when f_x, f_y, and f_z are all nonzero. Because the axial velocity solution is dependent on the solution of the Eulerian angles, the axial velocity solution is valid only when the Eulerian angle solution is valid.

By substituting Eq. (21) into Eq. (42), it can be shown that Eq. (42) simplifies to

$$A_{v11}(\tau_0, \tau; \lambda) = \bar{f}\phi(\tau_0) e^{(i\lambda\tau_0^2/2)} \bar{I}_{u0}(\tau_0, \tau; \lambda) + \lambda \bar{f} A_{v21}(\tau_0, \tau; \lambda, \rho) \tag{43}$$

where

$$A_{v21}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_\phi(\tau_0, u; \lambda, \rho)\, \mathrm{d}u \tag{44}$$

A_{v21} can be simplified after substituting Eqs. (24), (29), and (30) into Eq. (44) as follows:

$$A_{v21}(\tau_0,\tau;\lambda,\rho) = k_1 A_{v31}(\tau_0,\tau;\lambda,\rho) + k_2 A_{v32}(\tau_0,\tau;\kappa,\rho) \quad (45)$$

$$A_{v31}(\tau_0,\tau;\lambda,\rho) \triangleq \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{\phi 1}(\tau_0,u;\lambda,\rho)\,\mathrm{d}u \quad (46)$$

$$A_{v32}(\tau_0,\tau;\kappa,\rho) \triangleq \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{\phi 2}(\tau_0,u;\kappa,\rho)\,\mathrm{d}u \quad (47)$$

where A_{v31} and A_{v32} can be written as

$$A_{v31}(\tau_0,\tau;\lambda,\rho) = A_{v41}(\tau_0,\tau;\lambda,\rho) + A_{v42}(\tau_0,\tau;\lambda,\rho) \quad (48)$$

$$A_{v32}(\tau_0,\tau;\kappa,\rho) = A_{v43}(\tau_0,\tau;\kappa,\rho) + A_{v44}(\tau_0,\tau;\kappa,\rho) \quad (49)$$

and where

$$A_{v41}(\tau_0,\tau;\lambda,\mu) \triangleq [\Omega(\tau_0)e^{(-i\rho\tau_0^2/2)} - F\bar{I}_{u0}(\tau_0;\rho)] \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{u0}(\tau_0,u;\mu)\,\mathrm{d}u \quad (50)$$

$$A_{v42}(\tau_0,\tau;\lambda,\mu) \triangleq \int_{\tau_0}^{\tau} F e^{(-i\lambda u^2/2)} J_{u0}(\tau_0,u;\mu,\rho)\,\mathrm{d}u \quad (51)$$

and where

$$A_{v43}(\tau_0,\tau;\lambda,\kappa) \triangleq [\bar{\Omega}(\tau_0)e^{(i\rho\tau_0^2/2)} - \bar{F}I_{u0}(\tau_0;\rho)] \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{u0}(\tau_0,u;\kappa)\,\mathrm{d}u \quad (52)$$

$$A_{v44}(\tau_0,\tau;\lambda,\kappa) \triangleq \int_{\tau_0}^{\tau} \bar{F} e^{(-i\lambda u^2/2)} \bar{J}_{u0}(\tau_0,u;-\kappa,\rho)\,\mathrm{d}u \quad (53)$$

It can be shown that A_{v41} and A_{v42} can be written as

$$A_{v41}(\tau_0,\tau;\lambda,\mu) = [\Omega(\tau_0)e^{(-i\rho\tau_0^2/2)} - F\bar{I}_{u0}(\tau_0;\rho)][\bar{J}_{u0}(\tau_0,\tau;\lambda,\mu) - I_{u0}(\tau_0;\mu)\bar{I}_{u0}(\tau_0,\tau;\lambda)] \quad (54)$$

$$A_{v42}(\tau_0,\tau;\lambda,\mu) = A_{v51}(\tau;\mu,\rho) - FJ_{u0}(\tau_0;\mu,\rho)\bar{I}_{u0}(\tau_0,\tau;\lambda) \quad (55)$$

where

$$A_{v51}(\tau_0,\tau;\lambda,\mu) \triangleq \int_{\tau_0}^{\tau} F e^{(-i\lambda u^2/2)} J_{u0}(u;\mu,\rho)\,\mathrm{d}u \quad (56)$$

Also, A_{v43} and A_{v44} can be written as

$$A_{v43}(\tau_0,\tau;\lambda,\kappa) = [\bar{\Omega}(\tau_0)e^{(i\rho\tau_0^2/2)} - \bar{F}I_{u0}(\tau_0;\rho)][\bar{J}_{u0}(\tau_0,\tau;\lambda,\kappa) - I_{u0}(\tau_0;\kappa)\bar{I}_{u0}(\tau_0,\tau;\lambda)] \quad (57)$$

$$A_{v44}(\tau_0,\tau;\lambda,\kappa) = A_{v52}(\tau;\kappa,\rho) - \bar{F}\bar{J}_{u0}(\tau_0;-\kappa,\rho)\bar{I}_{u0}(\tau_0,\tau;\lambda) \quad (58)$$

where

$$A_{v52}(\tau_0,\tau;\lambda,\kappa) \triangleq \int_{\tau_0}^{\tau} \bar{F} e^{(-i\lambda u^2/2)} \bar{J}_{u0}(u;-\kappa,\rho)\,\mathrm{d}u \quad (59)$$

Using Appendix A, substituting Eq. (A1) into Eq. (56), we obtain

$$A_{v51}(\tau_s,\tau;\lambda,\mu) = A_{v61}(\tau;\lambda,\mu) + A_{v62}(\tau_s,\tau;\lambda,\mu) + FJ_{u0s}(\tau_s;\mu,\rho)\bar{I}_{u0}(\tau_0,\tau;\lambda) \quad (60)$$

where

$$A_{v61}(\tau;\lambda,\mu) \triangleq \int_{0}^{\tau} F e^{(-i\lambda u^2/2)} J_{u0s}(u;\mu,\rho)\,\mathrm{d}u, \qquad (\tau \le \tau_s) \quad (61)$$

$$A_{v62}(\tau_s,\tau;\lambda,\mu) \triangleq \int_{\tau_s}^{\tau} F e^{(-i\lambda u^2/2)} J_{u0l}(u;\mu,\rho)\,\mathrm{d}u, \qquad (\tau > \tau_s) \quad (62)$$

Similar to A_{v51}, A_{v52} can be written as

$$A_{v52}(\tau_s,\tau;\lambda,\kappa) = A_{v63}(\tau;\lambda,\kappa) + A_{v64}(\tau_s,\tau;\lambda,\kappa) + \bar{F}\bar{J}_{u0s}(\tau_s;-\kappa,\rho)\bar{I}_{u0}(\tau_0,\tau;\lambda) \quad (63)$$

where

$$A_{v63}(\tau;\lambda,\kappa) \triangleq \int_{0}^{\tau} \bar{F} e^{(-i\lambda u^2/2)} \bar{J}_{u0s}(u;-\kappa,\rho)\,\mathrm{d}u, \qquad (\tau \le \tau_s) \quad (64)$$

$$A_{v64}(\tau_s,\tau;\lambda,\kappa) \triangleq \int_{\tau_s}^{\tau} \bar{F} e^{(-i\lambda u^2/2)} \bar{J}_{u0l}(u;-\kappa,\rho)\,\mathrm{d}u, \qquad (\tau > \tau_s) \quad (65)$$

The functions A_{v61} and A_{v62} can be determined by substituting Eqs. (A2) and (A3) into Eqs. (61) and (62), respectively, as follows:

$$A_{v61}(\tau;\lambda,\mu) = \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (a_n + ib_n) \left(\frac{|\rho|}{8}\right)^{(n+\frac{1}{2})} \bar{J}_u(\tau;\lambda,2n+1) \quad (66)$$

$$J_u(\tau;\lambda,n) \triangleq \int_0^{\tau} e^{(i\lambda u^2/2)} \bar{I}_u(u;\lambda,n)\,\mathrm{d}u \qquad (n = 0,\ldots,11) \quad (67)$$

and

$$A_{v62}(\tau_s,\tau;\lambda,\mu) = A_{v71}(\tau_s,\tau;\lambda,\mu) + A_{v72}(\tau_s,\tau;\lambda,\rho) \quad (68)$$

where A_{v71} and A_{v72} are defined and simplified as

$$A_{v71}(\tau_s,\tau;\lambda,\mu) \triangleq F\sqrt{\frac{\pi}{|\rho|}}\frac{(1-i)}{2} \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)} I_{u0}(\tau_s,u;\mu)\,\mathrm{d}u = F\sqrt{\frac{\pi}{|\rho|}}\frac{(1-i)}{2}[\bar{J}_{u0}(\tau_s,\tau;\lambda,\mu) - I_{u0}(\tau_s;\mu)\bar{I}_{u0}(\tau_s,\tau;\lambda)] \quad (69)$$

$$A_{v72}(\tau_s,\tau;\lambda,\rho) \triangleq F\sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + id_n) \left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})} \times \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)} I_d(\tau_s,u;\lambda,2n+1)\,\mathrm{d}u = F\sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + id_n) \left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})} [I_d(\tau_s;\lambda,2n+1)\bar{I}_{u0}(\tau_s,\tau;\lambda) - J_d(\tau_s,\tau;\lambda,2n+1)] \quad (70)$$

and

$$J_d(\tau_s,\tau;\lambda,n) \triangleq \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)} I_d(u;\lambda,n)\,\mathrm{d}u \qquad (n = 0,\ldots,11) \quad (71)$$

In Appendices C, D, and E we provide details on the evaluation of $J_u(\tau;\lambda,n)$, $I_d(\tau_s,\tau;\lambda,n)$, and $J_d(\tau_s,\tau;\lambda,n)$, respectively.

It can be shown that A_{v63} and A_{v64} are obtained in a similar fashion to A_{v61} and A_{v62}:

$$A_{v63}(\tau;\lambda,\kappa) = \bar{F}\sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (a_n - ib_n) \left(\frac{|\rho|}{8}\right)^{(n+\frac{1}{2})} \bar{J}_u(\tau;\lambda,2n+1) \quad (72)$$

$$A_{v64}(\tau_s,\tau;\lambda,\kappa) = A_{v73}(\tau_s,\tau;\lambda,\kappa) + A_{v74}(\tau_s,\tau;\lambda,\rho) \quad (73)$$

$$A_{v73}(\tau_s,\tau;\lambda,\kappa) \triangleq \bar{F}\sqrt{\frac{\pi}{|\rho|}}\frac{(1+i)}{2}\int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)}\bar{I}_{u0}(\tau_s,u;-\kappa)\,\mathrm{d}u$$
$$= \bar{F}\sqrt{\frac{\pi}{|\rho|}}\frac{(1+i)}{2}[\bar{J}_{u0}(\tau_s,\tau;\lambda,\kappa) - I_{u0}(\tau_s;\kappa)\bar{I}_{u0}(\tau_s,\tau;\lambda)] \tag{74}$$

$$A_{v74}(\tau_s,\tau;\lambda,\rho) \triangleq \bar{F}\sqrt{\frac{\pi}{|\rho|}}\sum_{n=0}^{11}(c_n - id_n)\left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})}$$
$$\times \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)} I_d(\tau_s,u;\lambda,2n+1)\,\mathrm{d}u$$
$$= \bar{F}\sqrt{\frac{\pi}{|\rho|}}\sum_{n=0}^{11}(c_n - id_n)\left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})}[I_d(\tau_s;\lambda,2n+1)\bar{I}_{u0}(\tau_s,\tau;\lambda)$$
$$- J_d(\tau_s,\tau;\lambda,2n+1)] \tag{75}$$

By finding the functions A_{v71}, A_{v72}, A_{v73}, and A_{v74}, the closed-form analytical solution for the axial velocity problem is completed.

Asymptotic Limits for Special Cases

The axial velocity of a spinning-up rigid body when $f_z \neq 0$ is driven by the second term in Eq. (41). The problem is challenging when $f_x, f_y \neq 0$, and $f_z = 0$. Ayoubi and Longuski [36] and Ayoubi [37] showed that for an axisymmetric rigid body, when $f_x, f_y \neq 0$, and $f_z = 0$, the upper bound of the magnitude of axial velocity is given by

$$\Delta v_Z \leq \frac{|\lambda f|}{m}\left\{\frac{|\phi(\tau_0)|}{|\lambda|\tau_0} + \frac{|\Omega(\tau_0)|}{\sqrt{|k|}|\rho|\tau_0^2}\right.$$
$$\left. + \left[2\tau_0 + |\mu|\sqrt{\frac{\pi}{|\rho|}}\left(\frac{1}{|\rho|}+\frac{2}{|\mu|}\right)\left|\tilde{E}\left(\sqrt{\frac{|\rho|}{\pi}}\tau_0\right)\right|\right]\frac{\sqrt{M_x^2+M_y^2}}{I_z\tau_0^2}\right\} \tag{76}$$

When the initial conditions are zero (i.e., $|\phi(\tau_0)| = |\Omega(\tau_0)| = 0$), Eq. (76) simplifies to

$$|\Delta v_Z| \leq \frac{|\lambda f|}{m}\left[2\tau_0 + |\mu|\sqrt{\frac{\pi}{|\rho|}}\left(\frac{1}{|\rho|}+\frac{2}{|\mu|}\right)\left|\tilde{E}\left(\sqrt{\frac{|\rho|}{\pi}}\tau_0\right)\right|\right]$$
$$\times \frac{\sqrt{M_x^2+M_y^2}}{I_z\tau_0^2} \tag{77}$$

Figure 3 shows the contribution of the constant transverse forces, f_x and f_y, on $|\Delta v_Z|$ for a typical axisymmetric spacecraft [37]. For the very beginning of the spinning-up motion, the exact solution follows the "constant-spin limit" which is given by Ayoubi [37] as

$$\Delta v_{Z\,\mathrm{sec}} = \frac{(1+k)}{k}\frac{(f_y M_x - f_x M_y)t}{m I_z \omega_{z0}^2} \tag{78}$$

For large spin rates, the exact solution approaches the limit which is predicted by Eq. (77).

Now let us consider some geometric limits for a sphere, a thin rod, and a flat disk. It is shown in [36] that for a sphere, a thin rod, and a flat disk, $k \to 0, -1, +1$, respectively.

1) *Sphere* ($k \to 0$)

We recall that $\rho = \lambda k_{xy}$, when $k_{xy} \to 0$ (because $k \to 0$) then $\rho \to 0$. Also $\mu = \lambda$, thus we have

$$\lim_{\rho\to 0}\frac{\tilde{E}(\sqrt{\frac{|\rho|}{\pi}}\tau_0)}{\sqrt{\frac{|\rho|}{\pi}}} = \tau_0 \tag{79}$$

We note that the second term on the right-hand side of Eq. (77) grows without bound. Therefore

$$\Delta v_Z \to \infty \tag{80}$$

It is interesting to note that for a sphere, $v_z \to \infty$. For a sphere, we have $I_x = I_y = I_z$ and Euler's equations of motion result in

$$\omega_x(t) = M_x/I_x t + \omega_x(0) \tag{81}$$

$$\omega_y(t) = M_y/I_y t + \omega_y(0) \tag{82}$$

$$\omega_z(t) = M_z/I_z t + \omega_z(0) \tag{83}$$

which show that angular velocities change linearly with time. To get a better insight about the behavior of Δv_Z, we need to look at the time history of the Eulerian angles. Figure 4 shows the two Eulerian angles ϕ_x and ϕ_y for a typical sphere. We see that these angles remain nearly constant as $t \to \infty$. Therefore, the projections of the transverse forces f_x and f_y along the z axis are constant. (Here we recall that f_z does not have any contribution in the z direction because $f_z = 0$.)

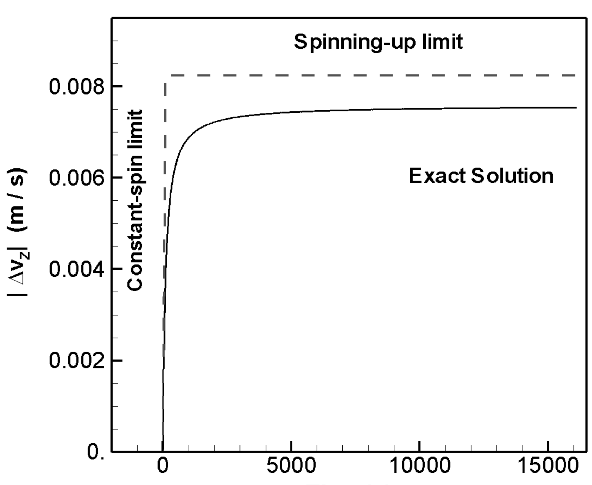

Fig. 3 Exact solution and asymptotic limits of the magnitude of the axial velocity, $|\Delta v_Z|$, when $f_x, f_y \neq 0$, and $f_z = 0$.

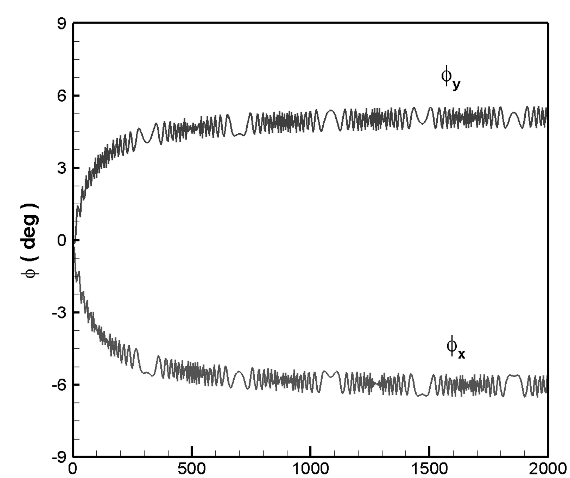

Fig. 4 Asymptotic behavior of the Eulerian angles ϕ_x and ϕ_y for a typical sphere.

2) *Thin rod* ($k \to -1$)

In this case, $\mu \to 0$, and $\rho \to \lambda$, so the limit of Eq. (77) is

$$|\Delta v_Z| \leq \frac{|\lambda f|}{m} \left\{ \frac{|\phi(\tau_0)|}{|\lambda|\tau_0} + \frac{\sqrt{\omega_x^2(\tau_0) + \omega_y^2(\tau_0)}}{|\lambda|\tau_0^2} + 2\left[\tau_0 + \sqrt{\frac{\pi}{|\lambda|}}\left|\tilde{E}\left(\sqrt{\frac{|\lambda|}{\pi}}\tau_0\right)\right|\right] \frac{\sqrt{M_x^2 + M_y^2}}{I_z \tau_0^2} \right\} \tag{84}$$

When $|\phi(\tau_0)| = |\omega(\tau_0)| = 0$, Eq. (84) reduces to

$$|\Delta v_Z| \leq \frac{2|\lambda f|}{m} \left[\tau_0 + \sqrt{\frac{\pi}{|\lambda|}}\left|\tilde{E}\left(\sqrt{\frac{|\lambda|}{\pi}}\tau_0\right)\right|\right] \frac{\sqrt{M_x^2 + M_y^2}}{I_z \tau_0^2} \tag{85}$$

3) *Flat disk* ($k \to +1$)

For a flat disk, we have $\mu \to 2\lambda$, and $\rho \to \lambda$. Thus, the limit of Eq. (77) is

$$|\Delta v_Z| \leq \frac{|\lambda f|}{m} \left\{ \frac{|\phi(\tau_0)|}{|\lambda|\tau_0} + \frac{\sqrt{\omega_x^2(\tau_0) + \omega_y^2(\tau_0)}}{|\lambda|\tau_0^2} + 2\left[\tau_0 + 2\sqrt{\frac{\pi}{|\lambda|}}\left|\tilde{E}\left(\sqrt{\frac{|\lambda|}{\pi}}\tau_0\right)\right|\right] \frac{\sqrt{M_x^2 + M_y^2}}{I_z \tau_0^2} \right\} \tag{86}$$

For zero initial conditions, that is, $|\phi(\tau_0)| = |\omega(\tau_0)| = 0$, we have

$$|\Delta v_Z| \leq \frac{2|\lambda f|}{m} \left[\tau_0 + 2\sqrt{\frac{\pi}{|\lambda|}}\left|\tilde{E}\left(\sqrt{\frac{|\lambda|}{\pi}}\tau_0\right)\right|\right] \frac{\sqrt{M_x^2 + M_y^2}}{I_z \tau_0^2} \tag{87}$$

Simulation and Numerical Results

In the simulations that follow, we define "exact solution" to represent a highly accurate numerical integration of Euler's equations of motion, the kinematic equations, and the inertial acceleration equations, Eqs. (32) and (33). MATHEMATICA® [40] is employed in this simulation to generate the exact solution. (We use the Bogacki–Shampine order-five method with local relative and absolute errors of order 10^{-14}.) The results are presented for the "low spin rate" and the "high spin rate" because of the approximation of the $J_{u0}(\tau_0, \tau; \mu, \rho)$ function via two piecewise-continuous functions (as shown in Appendix A). The low spin rate theory covers spin rates from 0 to 2.36 rpm and the high spin rate theory covers spin rates higher than 2.36 rpm. It is important to note that the axial velocity solution is valid when the Eulerian angles ϕ_x and ϕ_y are small.

The Galileo spacecraft mass properties and body-fixed forces and moments are used during the simulation

$$m = 2000 \text{ kg}, \qquad I_x = 2985$$
$$I_y = 2729, \qquad I_z = 4183 \text{ kg} \cdot \text{m}^2 \tag{88}$$

$$f_x = 7.66, \qquad f_y = -6.42, \qquad f_z = 10.0 \text{ N} \tag{89}$$

$$M_x = -1.253, \qquad M_y = -1.494, \qquad M_z = 13.5 \text{ N} \cdot \text{m} \tag{90}$$

with two sets of initial conditions as

$$\omega_x(0) = \omega_y(0) = \phi_x(0) = \phi_y(0) = \phi_z(0) = v_X(0) = v_Y(0) = v_Z(0) = 0, \quad w_z(0) = 0 \tag{91}$$

for low spin rate and

$$\omega_x(0) = \omega_y(0) = \phi_x(0) = \phi_y(0) = \phi_z(0) = v_X(0) = v_Y(0) = v_Z(0) = 0, \quad w_z(0) = 2.36 \text{ rpm} \tag{92}$$

for high spin rate.

Figures 5 and 6 show the axial velocity (Z component) for low and high spin rates in the inertial reference frame. The solid and dashed lines represent the exact and analytical solutions, respectively. As we expect, due to low gyroscopic rigidity, there is a noticeable difference between the exact and numerical solutions in the low spin case. (The Euler angles ϕ_x and ϕ_y reach maximum values of about 18 deg, which contributes to the small but noticeable errors observed in this simulation.)

The associated error between the exact and analytical solutions of the axial velocity is shown for low and high spin rates in Figs. 7 and 8, respectively. We see that the axial velocity relative error for low spin rate is less than 4%; for high spin rate it is less than 2% (for most of the duration).

We have tested our axial velocity solution for the cases when ω_{z0} and ω_{zf} are chosen in such a way that the combination of low spin rate and high spin rate theories are required. The results in accuracy are consistent with the accuracy of the results of the low and high spin rates which we have presented here.

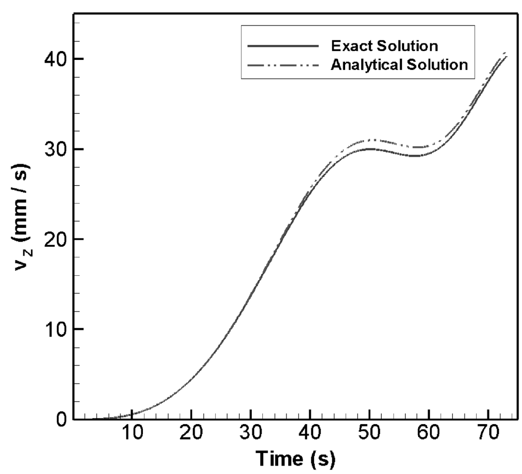

Fig. 5 Exact (solid line) and analytical (dash-dotted line) solutions for inertial axial velocity v_Z at a low spin rate ($\omega_{z0} = 0$, $\omega_{zf} = 2.36$ rpm).

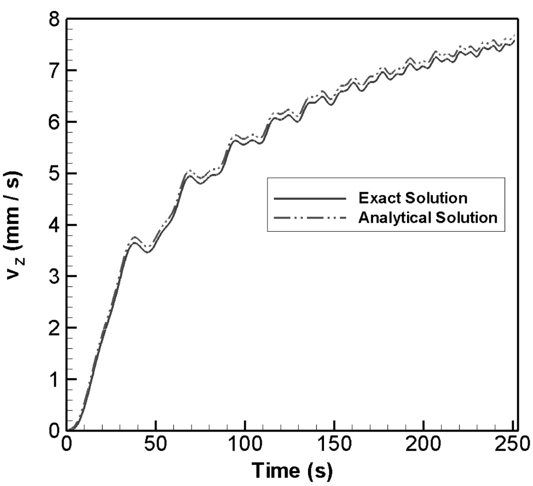

Fig. 6 Exact (solid line) and analytical (dash-dotted line) solution of inertial axial velocity v_Z at a high spin rate ($\omega_{z0} = 2.36$ rpm, $\omega_{zf} = 10$ rpm).

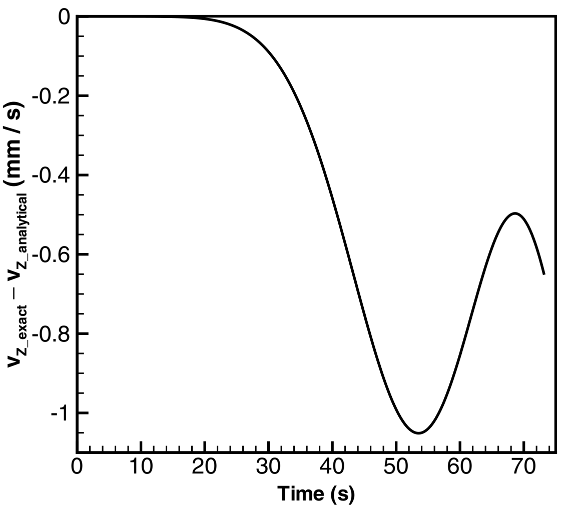

Fig. 7 Exact minus analytical solution of inertial axial velocity v_Z at a low spin rate ($\omega_{z0} = 0$, $\omega_{zf} = 2.36$ rpm).

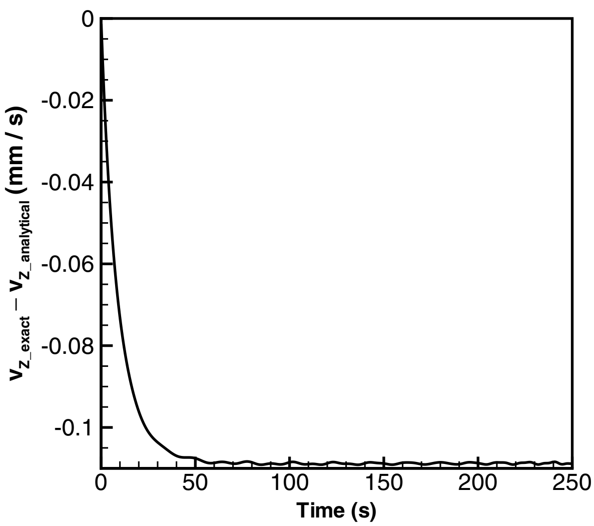

Fig. 8 Exact minus analytical solution of inertial axial velocity v_Z at a high spin rate ($\omega_{z0} = 2.36$ rpm, $\omega_{zf} = 10$ rpm).

Conclusions

We present a complete, closed-form, approximate analytical solution for the axial velocity of a spinning-up rigid body subject to constant body-fixed forces and torques about all three body axes. We use our analytical solutions to find an upper bound for the magnitude of the axial velocity. These bounds are derived for large spin rates and for some geometric limiting cases such as a sphere, a thin rod, and a flat disk. The axial velocity solution is valid for axisymmetric, nearly axisymmetric, and (under special conditions) asymmetric rigid bodies, when the Eulerian angles remain small. The solution consists of Fresnel integrals, integrals of the Fresnel integrals, and a new family of related integrals. The accuracy of the analytical solution has been verified by comparing with a highly precise numerical integration of the exact dynamical and kinematic equations. Applications of this analytical solution for axial velocity combined with the analytical theories developed in the literature (for transverse velocity and displacement) may include spacecraft on-board computations, probabilistic error modeling for mission planning, development of new control concepts for spacecraft maneuvers, and maneuver analysis.

Appendix A: $J_{u0}(\tau_0, u; \mu, \rho)$ Function

In the definition of $J_{u0}(\tau_0, u; \mu, \rho)$, Eq. (31), it was shown [41] that J_{u0} can be approximated as

Table A1 Boersma's numerical values of coefficients for the $J_{uo}(\tau; \mu, \rho)$ approximation [41]

i	a_i	b_i	c_i	d_i
0	1.595769140	−0.000000033	0.000000000	0.199471140
1	−0.000001702	4.255387524	−0.024933975	0.000000023
2	−6.808568854	−0.000092810	0.000003936	−0.009351341
3	−0.000576361	−7.780020400	0.005770956	0.000023006
4	6.920691902	−0.009520895	0.000689892	0.004851466
5	−0.016898657	5.075161298	−0.009497136	0.001903218
6	−3.050485660	−0.138341947	0.011948809	−0.017122914
7	−0.075752419	−1.363729124	−0.006748873	0.029064067
8	0.850663781	−0.403349276	0.000246420	−0.027928955
9	−0.025639041	0.702222016	0.002102967	0.016497308
10	−0.150230960	−0.216195929	−0.001217930	−0.005598515
11	0.034404779	0.019547031	0.000233939	0.000838386

$$J_{u0}(\tau_0, \tau; \mu, \rho) = \begin{cases} J_{u0s}(\tau; \mu, \rho), & \text{if } \tau \le \tau_s = \sqrt{8/|\rho|} \\ J_{u0s}(\tau_s; \mu, \rho) + J_{u0l}(\tau_s, \tau; \mu, \rho), & \text{otherwise} \end{cases} \tag{A1}$$

where, for $\tau \le \tau_s$,

$$J_{u0s}(u; \mu, \rho) \triangleq \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (a_n + ib_n) \left(\frac{|\rho|}{8}\right)^{(n+\frac{1}{2})} I_u(\tau; \lambda, 2n+1) \tag{A2}$$

and for $\tau > \tau_s$, we have

$$J_{u0l}(\tau_s, \tau; \mu, \rho) \triangleq \sqrt{\frac{\pi}{|\rho|}} \frac{(1-i)}{2} \int_{\tau_s}^{\tau} e^{(i\mu\xi^2/2)}\, d\xi + \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + id_n) \left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})} I_d(\tau_s, \tau; \lambda, 2n+1) \tag{A3}$$

(See, also, Table A1.) In Appendices B and D we provide more details on the evaluation of the $I_u(\tau; \lambda, n)$ and $I_d(\tau_s, \tau; \lambda, n)$ functions.

Appendix B: $I_u(u; \lambda, n)$ Function

In Appendix A, we defined a new function $I_u(u; \lambda, n)$ as follows:

$$I_u(\tau_0, \tau; \lambda, n) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} u^n\, du \tag{B1}$$

Using integration by parts, we find the recursive formula for $I_u(u; \lambda, n)$

$$I_u(\tau; \lambda, n) = \frac{-i\tau^{n-1}}{\lambda} e^{(i\lambda\tau^2/2)} + \frac{i(n-1)}{\lambda} I_u(\tau; \lambda, n-2) \qquad (n \ge 2) \tag{B2}$$

$$I_u(\tau; \lambda, 1) = \frac{-i}{\lambda}[e^{(i\lambda\tau^2/2)} - 1] \tag{B3}$$

$$I_{u0}(\tau; \lambda) = \sqrt{\frac{\pi}{|\lambda|}} \operatorname{sgn}(\tau) \tilde{E}(\sqrt{|\lambda|/\pi}\tau) \tag{B4}$$

where

$$\tilde{E}(\sqrt{|\lambda|/\pi}\tau) = \begin{cases} E(\sqrt{|\lambda|/\pi}\tau) & \text{when } \lambda < 0 \\ \bar{E}(\sqrt{|\lambda|/\pi}\tau) & \text{when } \lambda > 0 \end{cases} \tag{B5}$$

and

$$E(x) \triangleq \int_0^x e^{(-i\pi u^2/2)}\,du \tag{B6}$$

is the complex Fresnel integral. The sgn(·) symbol in Eq. (B4) represents the signum function which is $\mathrm{sgn}(x) = 1$ for $x \geq 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$.

$$I_u(\tau; 0, n) = \frac{\tau^{n+1}}{n+1} \tag{B7}$$

$$I_{u0}(\tau; 0) = \frac{\tau^2}{2} \tag{B8}$$

It can be easily shown that the integral of the $I_u(u; \lambda, n)$ satisfies the following recursive integral equation:

$$\int_0^\tau I_u(\xi; \lambda, n)\,d\xi = \frac{-i}{\lambda} I_u(\tau; \lambda, n-1) + \frac{i(n-1)}{\lambda}\int_0^\tau I_u(\xi; \lambda, n-2)\,d\xi \qquad (n \geq 2) \tag{B9}$$

where

$$\int_0^\tau I_{u0}(\xi; \lambda)\,d\xi = \sqrt{\frac{\pi}{|\lambda|}}\int_0^\tau \mathrm{sgn}(\xi)\tilde{E}(\sqrt{|\lambda|/\pi}\xi)\,d\xi \tag{B10}$$

$$\int_0^\tau I_u(\xi; \lambda, 1)\,d\xi = \int_0^\tau \frac{-i}{\lambda}[e^{(i\lambda\xi^2/2)} - 1]\,d\xi = \frac{-i}{\lambda}[\tau - I_{u0}(\tau; \lambda)] \tag{B11}$$

Appendix C: $J_u(u; \lambda, n)$ Function

Let us define

$$J_u(\tau_0, \tau; \mu, \rho, n) \triangleq \int_{\tau_0}^\tau e^{(i\mu u^2/2)} I_u(u; \rho, n)\,du = J_u(\tau; \mu, \rho, n) - J_u(\tau_0; \mu, \rho, n) \tag{C1}$$

For a special case, when μ and ρ are replaced with λ, the function (C1) reduces to $J_u(\tau; \lambda, n)$. It can be shown that

$$J_u(\tau_0, \tau; \mu, \rho, n) = \frac{i}{\rho} I_u(\tau_0, \tau; \rho - \mu, n-1) - i\frac{n-1}{\rho} J_u(\tau_0, \tau; \mu, \rho, n-2) \qquad (n \geq 2) \tag{C2}$$

where

$$J_u(\tau_0, \tau; \mu, \rho, 1) = \frac{i}{\rho}[I_{u0}(\tau_0, \tau; \lambda) - I_{u0}(\tau_0, \tau; \mu)] \tag{C3}$$

Appendix D: $I_d(u; \lambda, n)$ Function

In Appendix A, we defined another new function $I_d(u; \lambda, n)$ as follows:

$$I_d(\tau_0, \tau; \lambda, n) \triangleq \int_{\tau_0}^\tau \frac{e^{(i\lambda u^2/2)}}{u^n}\,du \tag{D1}$$

By integration by parts, we can show that

$$I_d(\tau; \lambda, n) = \begin{cases} \frac{e^{(i\lambda\tau^2/2)}}{(n-1)\tau^{(n-1)}} + \frac{i\lambda}{n-1} I_d(\tau; \lambda, n-2), & (n \geq 2; \lambda \neq 0) \\ \frac{1}{(n-1)\tau^{(n-1)}}, & (n \geq 2; \lambda = 0) \end{cases} \tag{D2}$$

where

$$I_{d0}(\tau; \lambda) = \begin{cases} I_{u0}(\infty; \lambda) - I_{u0}(\tau; \lambda), & (\lambda \neq 0) \\ -\tau, & (\lambda = 0) \end{cases} \tag{D3}$$

$$I_d(\tau; \lambda, 1) = \frac{1}{2} E_i\left(\frac{|\lambda|\tau^2}{2}\right) \tag{D4}$$

and $E_i(\tau)$ is the exponential integral function and is defined as

$$E_i(\tau) \triangleq \int_\tau^\infty \frac{e^{i\xi}}{\xi}\,d\xi \tag{D5}$$

A recursive integral formula can be found for the integral of the $I_d(u; \lambda, n)$ function as

$$\int_{\tau_s}^\tau I_d(\xi; \lambda, n)\,d\xi = \frac{1}{n-1}\left[I_d(\tau_s, \tau; \lambda, n-1) + i\lambda\int_{\tau_s}^\tau I_d(\xi; \lambda, n-2)\,d\xi\right] \qquad (n \geq 2) \tag{D6}$$

where

$$\int_{\tau_s}^\tau I_{d0}(\xi; \lambda)\,d\xi = I_{u0}(\infty; \lambda)(\tau - \tau_s) - \sqrt{\frac{\pi}{|\lambda|}}\int_{\tau_0}^\tau \mathrm{sgn}(\xi)\tilde{E}(\sqrt{|\lambda|/\pi}\xi)\,d\xi \tag{D7}$$

$$\int_{\tau_s}^\tau I_d(\xi; \lambda, 1)\,d\xi = \tau I_d(\tau; \lambda, 1) - \tau_s I_d(\tau_s; \lambda, 1) + I_{u0}(\tau_s, \tau; \lambda) \tag{D8}$$

Appendix E: $J_d(u; \lambda, n)$ Function

We define $J_d(u; \lambda, n)$ as

$$J_d(\tau_s, \tau; \mu, \rho, n) \triangleq \int_{\tau_s}^\tau e^{(-i\mu u^2/2)} I_d(u; \rho, n)\,du \tag{E1}$$

Equation (71) is a special case of Eq. (E1) when $\mu = \rho = \lambda$. We can show that the recursive formula for this function is given by

$$J_d(\tau_0, \tau; \mu, \rho, n) = \frac{1}{(n-1)(n-2)}\left[\frac{1}{\tau_s^{(n-2)}} - \frac{1}{\tau^{(n-2)}}\right] + \frac{i\lambda}{n-1} J_d(\tau_s, \tau; \mu, \rho, n-2) \qquad (n \geq 2) \tag{E2}$$

where

$$J_d(\tau_0, \tau; \mu, \rho, 0) = I_{u0}(\infty; \lambda)\bar{I}_{u0}(\tau_s, \tau; \lambda) - \bar{J}_{u0}(\tau_s, \tau; \mu, \rho) \tag{E3}$$

In general, determining $J_d(\tau_s, \tau; \mu, \rho, 1)$ is laborious; in addition, we need $J_d(\tau_s, \tau; \lambda, 1)$. In the following, we introduce an asymptotic expansion for the function $J_d(\tau_s, \tau; \lambda, 1)$. Recall the $J_d(\tau_s, \tau; \lambda, 1)$ definition which is

$$J_d(\tau_s, \tau; \lambda, 1) \triangleq \int_{\tau_s}^\tau e^{(-i\lambda u^2/2)} I_d(u; \lambda, 1)\,du \tag{E4}$$

By substituting Eq. (D4) into Eq. (E5), we obtain

$$J_d(\tau_s, \tau; \lambda, 1) = \frac{1}{2}\int_{\tau_s}^\tau e^{(-i\lambda u^2/2)} E_i\left(\frac{\lambda u^2}{2}\right)du \tag{E5}$$

Using integration by parts, we can show that $E_i(\lambda u^2/2)$ has an asymptotic expansion which is

$$E_i\left(\frac{\lambda u^2}{2}\right) \sim \sum_{n=0}^\infty \frac{-n! e^{(-i\lambda u^2/2)}}{(i\lambda u^2/2)^{(n+1)}} \tag{E6}$$

After substituting Eq. (E6) into Eq. (E5) and integrating, $J_d(\tau_s, \tau; \mu, \rho, 1)$ can be written as

$$J_d(\tau_s, \tau; \mu, \rho, 1) \sim \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)(i\lambda)^{(n+1)}} \left[\frac{1}{\tau^{(2n+1)}} - \frac{1}{\tau_s^{(2n+1)}} \right] \tag{E7}$$

The expansion (E7) is very accurate; considering three terms of the series (E7), we can show that the order of magnitude of the relative error is about 10^{-6}.

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